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Bilinear Forms on Novikov Algebras

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Novikov algebras were introduced in connection with the Poisson brackets of hydrodynamic type and Hamiltonian operators in the formal variational calculus. On the other hand, there can be geometry and Lagrangian mechanics on homogenous spaces related to Novikov algebras. The nondegenerate symmetric bilinear forms on Novikov algebras can be regarded as the pseudometrics, and some additional identities for these forms correspond to some "conserved quantities." In particular, there is an important kind of "conserved" nondegenerate symmetric bilinear forms that correspond to the pseudo-Riemannian connections such that parallel translation preserves the bilinear form on the tangent spaces. Moreover, the fact that the left multiplication operators form a Lie algebra for a Novikov algebra is compatible with such a form. However, we show in this note that there are no such forms on most Novikov algebras in low dimensions.

KEY WORDS: Novikov algebra; transitive Novikov algebra; bilinear form; pseudometric; pseudo-Riemannian connection.

1. INTRODUCTION

Poisson brackets of the hydrodynamic type were introduced and studied in Dubrovin and Novikov (1983, 1984) and Balinskii and Novikov (1985):

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + \sum_{k=1}^{N} u_x^k b_k^{ij}(u(x))\delta(x-y). \tag{1.1}$$

The simplest local Lie algebra arising from brackets of hydrodynamic type (1.1) was also introduced as follows (Balinskii and Novikov, 1985):

$$g^{ij} = \sum_{k=1}^{n} C_k^{ij} u^k + g_0^{ij}, \quad b_k^{ij} = \text{const}, \; g_0^{ij} = \text{const}, \tag{1.2}$$

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$$\begin{aligned}[p, q]_k(z) &= b_k^{ij}(p_i(z)q_j'(z) - q_i(z)p_j'(z)),\\ b_k^{ij} + b_k^{ji} &= C_k^{ij} = \partial g^{ij}/\partial u^k.\end{aligned} \tag{1.3}$$

From the Jacobi identity, the tensor b_k^{ij} by Eq. (1.3) defines a local translationally invariant Lie algebra of first order if and only if $\{b_k^{ij}\}$ is the set of structure constants of a new finite-dimensional algebra A with a bilinear product $(x, y) \to xy$ satisfying

$$e_i e_j = \sum_{k=1}^{n} b_k^{ij} e_k, \tag{1.4}$$

$$(x, y, z) = (y, x, z), \tag{1.5}$$

$$(xy)z = (xz)y, \tag{1.6}$$

for any $x, y, z \in A$. Here $\{e_1, e_2, \ldots, e_n\}$ is a basis of A and $(x, y, z) = (xy)z - x(yz)$. (Note that we use the left symmetry here, whereas the right symmetry was used in Balinskii and Novikov (1985), Dubrovin and Novikov (1983, 1984), and Zel'manov (1987)).

The algebra A satisfying Eqs. (1.5)–(1.6) is called a "Novikov algebra" by Osborn (Osborn, 1992a,b, 1994; Xu, 1996, 1997, 2000). It also has a close connection to some Hamiltonian operators in the formal variational calculus (Gel'fand and Diki, 1975, 1976; Gel'fand and Dorfman, 1979; Xu, 1995) and some nonlinear partial differential equations, such as KdV equations (Dubrovin and Novikov, 1983; Gel'fand and Diki, 1975, 1976. On the other hand, Novikov algebras are a special class of left-symmetric algebras that only satisfy Eq. (1.5). Left-symmetric algebras are nonassociative algebras arising from the study of affine manifolds, affine structures, and convex homogeneous cones (Bai and Meng, 2000; Burde, 1998; Kim, 1986; Vinberg, 1963).

The commutator of a Novikov algebra (or a left-symmetric algebra) A,

$$[x, y] = xy - yx, \tag{1.7}$$

defines a Lie algebra $\mathcal{G} = \mathcal{G}(A)$. Let L_x and R_x denote the left and right multiplication respectively, i.e., $L_x(y) = xy$, $R_x(y) = yx$, $\forall x, y \in A$. Then for a Novikov algebra, the left multiplication operators form a Lie algebra and the right multiplication operators are commutative. If every R_x is nilpotent, then A is called right nilpotent or transitive. The transitivity corresponds to the completeness of the affine manifolds in geometry (Kim, 1986; Vinberg, 1963).

On the other hand, a theory for expressing the relationship between a Lagrangian mechanical system on a homogeneous space and its differential geometric properties in terms of nonassociative algebras is given in Sagle (1985) and Sagle and Walde (1973). In particular, if we take the considered nonassociative algebra as a Novikov algebra, we can obtain the corresponding geometry and

Lagrangian mechanics, although many concrete geometrical concepts and physical phenomena related to Novikov algebras are still unclear in some sense. One of the effective ways to solve such questions is to derive some algebraic properties of Novikov algebras as the first step. The study about the bilinear forms on Novikov algebras is a good choice. In fact, the pseudometric connections related to differential geometry and Lagrangian mechanics are decided by the nondegenerate bilinear forms on Novikov algebras, and some additional identities for these forms arise from some "conserved quantities." In Balinskii and Novikov (1985) and Zel'manov (1987), the authors studied the so-called invariant bilinear form such that every right multiplication operator is adjoint. In Sagle (1985), there is another important kind of "conserved" nondegenerate symmetric bilinear forms that correspond to the pseudo-Riemannian connections such that parallel translation preserves the bilinear form on the tangent spaces.

Furthermore, we can see that these two kinds of forms just correspond to the demands for right and left multiplication operators of Novikov algebras respectively. That is, for a Novikov algebra, the fact that the right multiplication operators are commutative is compatible with the invariant bilinear forms and the fact that that left multiplication operators form a Lie algebra is compatible with the pseudo-Riemannian connections. For the former, a more detailed discussion is given in Bai and Meng (2001c). For the latter, however, just as we will see in this note, the existence of such forms is in question. In particular, we can see that there are no pseudo-Riemannian connections on most of the Novikov algebras in low dimensions. This means Eqs. (1.5) and (1.6) have some intrinsic constraints for additional "invariance" for the nondegenerate symmetric bilinear forms on Novikov algebras. This note is organized as follows. In Section 2, we discuss the bilinear forms on Novikov algebras corresponding to the pseudo-Riemannian connection. In Section 3, we give some conclusion based on the discussion in the previous section.

2. PSEUDO-RIEMANNIAN CONNECTION

A pseudo-Riemannian connection is a pseudometric connection such that the torsion is zero, and parallel translation perseveres the bilinear form on the tangent spaces (Sagle, 1985). The corresponding structure on a Novikov algebra A is a nondegenerate symmetric bilinear form $f: A \times A \to \mathbf{F}$ such that

$$f(xy, z) + f(y, xz) = 0, \ \forall x, y, z \in A \tag{2.1}$$

Let $\{e_1, \ldots, e_n\}$ be a basis of A, then we have

$$f(e_i e_j, e_k) + f(e_j, e_i e_k) = 0. \tag{2.2}$$

Moreover a bilinear form on A under the basis $\{e_1, \ldots, e_n\}$ is completely decided by the matrix $\mathcal{F} = (f_{ij})$, where

$$f_{ij} = f(e_i, e_j). \tag{2.3}$$

The form is symmetric if and only if $\mathcal{F}$ is symmetric and the form is nondegenerate if and only if the determinant of $\mathcal{F}$ is not zero.

Let $\{c_k^{ij}\}$ be the set of structure constants of A, i.e.,

$$e_i e_j = \sum_k^n c_k^{ij} e_k. \tag{2.4}$$

Then by Eq. (2.2), we have

$$\sum_{l=1}^n c_l^{ij} f_{lk} + \sum_{l=1}^n c_l^{ik} f_{jl} = 0. \tag{2.5}$$

This means that f_{ij} can be solved directly through these homogeneous linear equations with the coefficients c_{ij}^k.

Furthermore, we can see that the fact that left multiplication operators form a Lie algebra is compatible with Eq. (2.1). In fact, for all $x, y, z, w \in A$, we have

$$f(L_x L_y(z), w) = -f(L_y(z), L_x(w)) = f(z, L_y L_x(w)),$$
$$f(L_y L_x(z), w) = -f(L_x(z), L_y(w)) = f(z, L_x L_y(w)).$$

Hence by the fact that left multiplication operators form a Lie algebra, the relation

$$f(L_x L_y(z) - L_y L_x(z), w) = f(z, L_y L_x(w) - L_x L_y(w))$$

is equivalent to the relation

$$f\left(L_{[x,y]}(z), w\right) = -f\left(z, L_{[x,y]}(w)\right),$$

which can be obtained from Eq. (2.1) directly.

However, the symmetric bilinear forms satisfying Eq. (2.1) may be degenerate. It is obvious that there is no nondegenerate symmetric bilinear form satisfying Eq. (2.1) on the one-dimensional nontrivial Novikov algebra (a Novikov algebra is called trivial if all structure constants c_k^{ij} are zero).

Example 2.1. Let us see the bilinear forms satisfying Eq. (2.1) on two-dimensional Novikov algebras over the complex number field the classification of which is given in Bai and Meng (2001a,b). Recall that the (form) characteristic matrix of a Novikov algebra is defined as

$$\mathcal{A} = \begin{pmatrix} \sum_{k=1}^n c_k^{11} e_k & \cdots & \sum_{k=1}^n c_k^{1n} e_k \\ \cdots & \cdots & \cdots \\ \sum_{k=1}^n c_k^{n1} e_k & \cdots & \sum_{k=1}^n c_k^{nn} e_k \end{pmatrix}, \tag{2.6}$$

Then we have the following table through Eq. (2.5).

Characteristic matrix	Bilinear forms satisfying Eq. (3.1)	Symmetric bilinear forms satisfying Eq. (3.1)	Determinant of symmetric forms
(T1) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{pmatrix}$	$f_{11} f_{22} - f_{12}^2$
(T2) $\begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} f_{11} & f_{12} \\ -f_{12} & 0 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} f_{11} & 0 \\ 0 & 0 \end{pmatrix}$	0
(T3) $\begin{pmatrix} 0 & 0 \\ -e_1 & 0 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & 0 \\ 0 & f_{22} \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & 0 \\ 0 & f_{22} \end{pmatrix}$	0
(N1) $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0
(N2) $\begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & 0 \\ 0 & f_{22} \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & 0 \\ 0 & f_{22} \end{pmatrix}$	0
(N3) $\begin{pmatrix} e_1 & e_2 \\ e_2 & 0 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0
(N4) $\begin{pmatrix} 0 & e_1 \\ 0 & e_2 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0
(N5) $\begin{pmatrix} 0 & e_1 \\ 0 & e_1 + e_2 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0
(N6) $\begin{pmatrix} 0 & e_1 \\ le_1 & e_2 \end{pmatrix}$ $l \neq 0, 1$	$\mathcal{F} = \begin{pmatrix} 0 & f_{12} \\ -f_{12} & 0 \end{pmatrix} l = -1$ $\mathcal{F} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} l \neq -1$	$\mathcal{F} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0

So we know that except (T1), which is the trivial Novikov algebra in two dimensions, the symmetric bilinear forms satisfying Eq. (2.1) on two-dimensional Novikov algebras must be degenerate.

In fact, the phenomenon appearing in Example 2.1 is not accidental. Any nontrivial Novikov algebra in two dimensions belongs to the following case:

Claim. Let A be a Novikov algebra with the structure constants c_k^{ij}. If there exist i, j, and k such that

$$c_k^{ij} = 1, \qquad c_{k'}^{ij'} = 0, \quad \forall j' \neq j,\ k' \neq k, \tag{2.7}$$

then any symmetric bilinear form satisfying Eq. (2.1) is degenerate.

In fact, Eq. (2.7) means that in the characteristic matrix (2.6) of A, there exists a row such that there is only one structure constant that is not zero (it is 1) in this row. By Eq. (2.5), we have

$$f_{kl} = c_k^{ij} f_{kl} = \sum_{m=1}^{n} c_m^{ij} f_{ml} = -\sum_{m=1}^{n} c_m^{il} f_{jm} = -c_k^{il} f_{jk}, \quad \forall l = 1, 2, \ldots, n.$$

If $l \neq j$, then we have $f_{kl} = 0$. If $l = j$, then we have $f_{kj} = -f_{jk}$. If the bilinear form is symmetric, then $f_{kj} = 0$. So in the matrix $\mathcal{F}$, the elements in kth row are zero. Thus the determinant of $\mathcal{F}$ is zero, i.e., the form f is degenerate.

Furthermore, from the classification of three-dimensional Novikov algebras over **C** given in Bai and Meng (2001a,b), we can see that all non-transitive Novikov algebras in three dimensions satisfying the conditions in the above claim. Hence there are no pseudo-Riemannian connections on any three-dimensional nontransitive Novikov algebras. For the transitive cases, we have

Example 2.2. There are following types of three-dimensional transitive Novikov algebras (Bai and Meng, 2001a,b) which do not belong to the cases in the above claim: (A1), (A6), (A8), (A10), (A11), and (A12). The symmetric bilinear forms satisfying Eq. (2.1) on these Novikov algebras are given in the following table:

Characteristic matrix	Symmetric bilinear forms satisfying Eq. (2.1)	Determinant of symmetric forms
(A1) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}$	$\det \mathcal{F}$
(A6) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & e_1 \\ 0 & -e_1 & le_1 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & f_{22} & f_{23} \\ 0 & f_{23} & f_{33} \end{pmatrix}$	0
(A8) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_1 & e_2 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f_{33} \end{pmatrix}$	0
(A10) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & e_1 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f_{33} \end{pmatrix}$	0
(A11) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & le_2 & 0 \end{pmatrix}$ $\lvert l \rvert \leq 1, l \neq 0$	$\mathcal{F} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f_{33} \end{pmatrix}$ $l \neq -1$	0
	$\mathcal{F} = \begin{pmatrix} 0 & f_{12} & 0 \\ f_{12} & 0 & 0 \\ 0 & 0 & f_{33} \end{pmatrix}$ $l = -1$	$-f_{33} f_{12}^2$
(A12) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & e_1 + e_2 & 0 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f_{33} \end{pmatrix}$	0

Hence, except the trivial Novikov algebra (A1) and the type (A11) with $l = -1$, there does not exist any nondegenerate symmetric bilinear form satisfying Eq. (2.1) on Novikov algebras in three dimensions.

3. CONCLUSION AND DISCUSSION

From the discussion on the previous section, we know that except the trivial Novikov algebras and the type (A11) with $l = -1$, there is no pseudo-Riemannian connection on any Novikov algebra in less than or equal to three dimensions. We believe that such a phenomenon can appear in higher dimensions. In some sense, this means that this kind of pseudometric is not very "suitable" for Novikov algebras, although it is very important and even seems "reasonable" for the demands for left multiplication operators. The similar situation can happen in other kinds of pseudometrics. For example, in Sagle (1985), there is another kind of pseudometric such that the tangent length of a geodesic is constant. The corresponding symmetric bilinear form satisfies $f(xx, x) = 0$ for every $x \in A$. On most of Novikov algebras in low dimensions, we find that such forms are also degenerate.

On the other hand, there will be some suitable bilinear forms for Novikov algebras. As in the introduction, one such kind of bilinear form is the invariant bilinear form (Bai and Meng, 2001a,b; Balinskii and Novikov, 1985; Zel'manov, 1987). Under such a form, the right multiplication operator is adjoint: $f(xy, z) = f(x, zy)$. Comparing with the cases in the study of pseudo-Riemannian connection, there exist much fewer Novikov algebras where there does not exist any nondegenerate symmetric invariant bilinear forms. On the contrary, in Bai and Meng (2001c), we can see that there exist nondegenerate symmetric invariant bilinear forms on most of Novikov algebras.

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